Math 391: Homework 2.0  
Fall 2016

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Due date: Sept 19, 11:55pm
Submission Instructions:

- Each group must submit one solution to this assignment in class on the due date.
- Assignments must be typed. They must be turned in on Moodle.
- Each group must submit only one solution. If multiple solutions are handed in, the grader will randomly discard all but one of them.
- Your submission must have the name of every group member clearly marked. If your name is not on it, you do not get credit.
- All arguments should be clear, concise, and (of course) correct. You will lose points for poor writing, bad grammar, etc...
- Use of the Internet for solving these problems is strictly forbidden, and will be treated as an honor code violation.
- Inter-group work: Members of different groups may discuss the questions. However:
  - With your submission you must provide a list of all non-group members with whom any member of your group discussed the assignment.
  - No written material may leave the inter-group discussion. If you talk with someone about the assignment, you must throw away and written notes at the end of the discussion.
Problem 1 (30 points): Queen Consolidated has developed a new inter-city transportation method to move products between its $n$ distribution centers scattered around Starling city. They have dug tunnels directly between every pair of centers. Tunnels never intersect, and are designated to carry goods in a single direction. So between any two centers $A$ and $B$, there is either a tunnel to carry goods from $A$ to $B$ or one to carry goods from $B$ to $A$ – but not both.

Prove that, regardless of the value of $n$ or how the tunnels are directed, there must be way to walk through tunnels in order to visit every distribution center without ever going the wrong direction in a tunnel.

Solution:
Proof by induction:

Base case ($n = 2$): If there $n = 2$ centers, there is only one tunnel. Clearly we can follow this tunnel from the source to the destination and hit ever center.

Induction assumption: Assume that for some $k \geq 2$, for any configuration of tunnels, there is a path that visits every center once.

Induction step: We now prove the statement is true for $k + 1$. Suppose we have our $k + 1$ centers. Pick some center $A$ and remove it. We are now left with $k$ centers, so by our assumption there is a path that visits all of them. Label the centers along this path $v_1, v_2, \ldots, v_k$ (in the order defined by that path), and let $i$ be the smallest value such that there is a tunnel from $A$ to $v_i$ (if such an $i$ exists). Now there are three possibilities:

- No such $i$ exists – all tunnels go from $v_i$ to $A$. Then the path we are looking for is $v_1, v_2, \ldots, v_k, A$.
- $i = 1$: there is a tunnel from $A$ to $v_1$. Then the path we are looking for is $A, v_1, \ldots, v_k$.
- $i$ exists and $i > 1$: Then there is tunnel from $A$ to $v_i$, and there is a tunnel from $v_{i-1}$ to $A$ (since $i$ is the smallest value where there is a tunnel from $A$ to $v_i$). So now the path we want is: $v_1, \ldots, v_{i-1}, A, v_i, \ldots, v_k$.

Regardless of the case, we can find the path we are looking for.

Comment: Notice that the induction proof also defines an algorithm. If you understand this proof, writing an algorithm to find the actual path should be easy.

Common mistakes in submissions:

1. You need to induct down, not up. That is, in the induction step, start with an instance of $n + 1$ centers, then remove a tunnel (as opposed to starting with an instance of $n$ tunnels and adding one). This saves you the bother of then having to prove that every single problem instance can be built from a smaller problem instance.

2. You need to use words, not just pictures, to explain your proof. Pictures can get you into trouble, because they often are more specific and can mean you are not addressing the general case.

3. Carefully read the problem description, and make sure you understand what the problem is asking (in this case, understanding how the distribution centers and tunnels are interconnected is very important)

Grading Rubric: The following is a rough guide to how this question was graded. However, as no two proofs are exactly the same, I reserve the right to deviate in order to account for alternative approaches, odd mistakes, lack of clarity in writing, or illegibility.

- Applying induction to the problem (6 pts):
  - 3 points for the base case.
– 3 points the assumption.

• Proof of correctness (12 pts):
  – 2 points for inducting down (or correctly inducting up – presenting all details).
  – 4 points for attaching the removed node to either end.
  – 6 points for inserting the removed node in the middle.

• Quality of proof (2 pts):
  – 1 point for organization of proof (is it easy to follow?)
  – 1 points for conciseness (i.e. no long winded explanations, no extra base cases, etc.)
Problem 2 (30 points): Let $G$ be a weighted, undirected, connected graph and consider the following definitions:

- A spanning tree $T$ of $G$ is a graph that:
  1. Uses every node of $G$.
  2. Uses only edges from $G$ (but not necessarily all edges).
  3. Is a tree (no cycles, connected).

Notice that spanning tree is not necessarily unique – there may be many different spanning trees.

- The weight of a spanning tree $T$ is the sum of the weights of all of the edges used in $T$.

- A minimum spanning tree (MST) of $G$ is a spanning tree of $G$ that has the minimum weight over all spanning trees of $G$. (That is, there is no spanning tree of $G$ which has a smaller weight.) Notice that the MST may also not be unique: there can be multiple MSTs for a graph. (Example: draw four points in a square pattern, and give all four edges the same weight. You will find four different MSTs.)

Let $G$ be a weighted graph such that no two edges have the same weight. Prove that $G$ has only one MST.

Hints:

- This will be done by contradiction, not induction.

- Your best strategy is to assume that there are two MSTs, and show you can then build a spanning tree that is smaller than the MSTs. (Contradiction.)

- What happens if you take an edge that is not used in both trees, and add it back into the tree not using it?

Solution:

Assume there are two different MSTs $T_1$ and $T_2$, and let $e$ be the edge with the smallest weight that is used in one tree but not both. ($e$ must exist, or the trees would be identical). Assume without loss of generality that $e$ is contained in $T_1$ (hence not in $T_2$). Now create a new tree $T_2'$ by adding $e$ to $T_2$, and notice that $T_2'$ must now contain a cycle. This is because $e$ connects two nodes that did not have an edge between them in $T_2$ (since $e$ was not in $T_2$), but were connected by a path in $T_2$ (since $T_2$ is a tree – hence must be connected). Thus there is a cycle consisting of that path plus $e$.

We have established that $T_2'$ has a cycle. We also know that at least one edge on that cycle is not used in $T_1$ (since $T_1$ is a tree), and that edge is not $e$ (since we picked $e$ because it was in $T_1$). Call this edge $e'$. We know that $w(e') \geq w(e)$ (since $e$ was the edge with the smallest weight not in both trees), an we know that $w(e') \neq w(e)$ (since the problem states that every edge has a unique length). Thus $w(e') > w(e)$ – edge $e'$ must have a larger weight than edge $e$.

Create a tree $T_2''$ by removing $e'$ from tree $T_2'$. Notice $T_2''$ must be a tree: since $e'$ was part of a cycle in $T_2'$, the new graph must remain connected (what happens when you remove one edge from a cycle?) and can no longer have any cycles (you cannot create a new cycle by removing an edge). Further, $T_2''$ uses only edges that are in $G$ ($T_2$ was a spanning tree, and $e$ was a member of another spanning tree – so all edges in $T_2'$ must be edges of the original graph.) So $T_2''$ is a spanning tree of $G$.

$T_2''$ was created by starting with $T_2$, adding new edge $e'$, and removing old edge $e''$, so the total weight of $T_2''$ is:

$$w(T_2'') = w(T_2) + w(e) - w(e')$$
$$= w(T_2) + (w(e) - w(e'))$$
$$< w(T_2) + 0, \text{ since } w(e) < w(e')$$
So we have: $T'_2$ is a spanning tree of the original graph $G$ whose total weight is less than that of the minimum spanning tree $T_2$ – which is, by definition, impossible. Hence we have a proof by contradiction: If we start with two different MSTs of $G$ and every edge of $G$ has a unique weight, then we can create a spanning tree of smaller weight than the MSTs. We conclude that the premise is impossible: we cannot have two different MSTs of a graph if the graph has only unique edge weights.

Common mistakes in submissions:

1. You need to give support for every claim you make. Do not assume that anything is a “given”.

2. Further, be careful to make sure that you are not making assumptions that are not given to you by the problem (and cannot be proven from what you were given).

3. You cannot claim that for any two edges $e, f$ where WLOG $w(e) < w(f)$ ($e \in T_1$ and $f \in T_2$) putting $e$ into $T_2$ will allow you to remove $f$ from $T_2$ to create a smaller weight tree. You need to somehow prove that there is a relationship between $e$ and $f$. Otherwise, you might have ended up adding $e$, creating a cycle in $T'_2$, and removing $f$, disconnecting $T'_2$ into two connected components (one of which still has that cycle).

4. You might try to claim that this is inferred simply by the fact that each edge has a different length, but the knowledge that edge weights are unique does not implicitly prove that two different spanning trees will have a different overall weight. Even though they contain different edges, the summations of their edge weights can still be equal.

5. A specific example of a particular graph does not prove anything about the general case. In order to prove that there is only one MST for all graphs with unique edge weights, you need to do a general proof.

Grading Rubric: The following is a rough guide to how this question was graded. However, as no two proofs are exactly the same, I reserve the right to deviate in order to account for alternative approaches, odd mistakes, lack of clarity in writing, or illegibility.

1. If you attempted a proof by contradiction as per the suggestions:
   - Taking hint (5 pts)
     - Proof by contradiction (assume there at two MSTs)
     - Adding an edge.
   - Details (15 pts)
     - 3 points for edge selection (selecting smallest edge $e$ in exactly 1 tree)
     - 4 points for edge insertion
       - 3 for making claim that inserting $e$ into other tree creates cycle $C$
       - 1 for providing support for claim
     - 4 points for edge removal
       - 3 for choosing edge $e'$ in cycle $C$ with larger weight than $e$ for removal
       - 1 for justifying existence of edge $e'$
     - 4 points for new MST and comparison
       - 2 for making/providing support for claim that after removing $e'$ we again have a spanning tree
       - 2 for wrapping up by showing that new tree weighs less than 2 previous

2. Otherwise, I graded on a case by case basis.